Fundamental structure of steady plastic shock waves in metals

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The propagation of steady plane shock waves in metallic materials is considered. Following the constitutive framework adopted by R. J. Clifton [Shock Waves and the Mechanical Properties of Solids, edited by J. J. Burke and V. Weiss (Syracuse University Press, Syracuse, N.Y., 1971), p. 73] for analyzing elastic–plastic transient waves, an analytical solution of the steady state propagation of plastic shocks is proposed. The problem is formulated in a Lagrangian setting appropriate for large deformations. The material response is characterized by a quasistatic tensile (compression) test (providing the isothermal strain hardening law). In addition the elastic response is determined up to second order elastic constants by ultrasonic measurements. Based on this simple information, it is shown that the shock kinetics can be quite well described for moderate shocks in aluminum with stress amplitude up to 10 GPa. Under the later assumption, the elastic response is assumed to be isentropic, and thermomechanical coupling is neglected. The model material considered here is aluminum, but the analysis is general and can be applied to any viscoplastic material subjected to moderate amplitude shocks. Comparisons with experimental data are made for the shock velocity, the particle velocity and the shock structure. The shock structure is obtained by quadrature of a first order differential equation, which provides analytical results under certain simplifying assumptions. The effects of material parameters and loading conditions on the shock kinetics and shock structure are discussed. The shock width is characterized by assuming an overstress formulation for the viscoplastic response. The effects on the shock structure of strain rate sensitivity are analyzed and the rationale for the J. W. Swegle and D. E. Grady [J. Appl. Phys. 58, 692 (1985)] universal scaling law for homogeneous materials is explored. Finally, the ability to deduce information on the viscoplastic response of materials subjected to very high strain rates from shock wave experiments is discussed. © 2004 American Institute of Physics. [DOI: 10.1063/1.1640452]

I. INTRODUCTION

The analysis of plastic shock waves has been the object of numerous studies starting in the years 1942–1945 coinciding with the Manhattan project. A review of this early work is given by Rice et al.1 Of particular interest are the plate impact experiments which provide valuable information on the response of materials upon compressive shock loading. A key result is the Hugoniot diagram, also called the equation of state which relates the compressive stress to the specific volume. Extensive studies have been conducted, and Hugoniot diagrams have been obtained for a large variety of materials (see Marsh).2

For a certain range of shock amplitudes, two compressive shock waves are generated in metallic materials. The first shock is the elastic precursor which compresses the material to the elastic limit. The state reached at the rear of the elastic precursor will be referred to as state (+) in the following. The elastic precursor is followed by the plastic wave, which further compresses the material to the final state, referred to as state (−). The transition from state (+) to state (−) occurs within a thin layer (the plastic shock front) where the material is rapidly deformed. For loadings of large amplitude, steady plastic shocks are observed. In this circumstance the shock front [where the transition from state (+) to state (−) occurs] propagates with a constant velocity and with a constant shape. These steady profiles result from the balance between front steepening due to non-linearity of the material response and smoothing due to dissipation. Steady wave shocks have been first analyzed in Newtonian fluids by Rayleigh3 and Taylor,4 and in viscoelastic solids by Band,5 Band and Duvall,6 and Bland.7 In solids, the role of viscous like effects on the shock structure (or profile) has been experimentally identified in the 1960s by Barker,8 Mineev and Savinov,9 Mineev and Zaidel,10 Johnson and Barker,11 Manvi et al.;12 see also Swan et al.,13 Prieto and Renero,14 and the summary on this subject by Godounov et al.15

Plastic shock structures have been analyzed, with the perspective of understanding how shock profiles are affected by the material viscous response. In particular, Johnson and Barker11 and Swegle and Grady16 have analyzed experimental recordings where the particle velocity at the rear of the plastic shock is reported in terms of time. They have discussed the effect of the strain rate sensitivity on steady plastic shock profiles. According to Bland,7 steady plastic shocks
can be observed when the propagation thickness is larger than five times the plastic shock front thickness. Swegle and Grady\textsuperscript{16} analyzed steady plastic shock fronts for a variety of materials and modeled the experiments by using Hugoniot diagrams reported in the literature. For the purpose of analyzing the shock structure, they assumed a specific form of the viscoplastic material response. The wave analysis is made with a finite difference numerical code. An important result is the relationship obtained between the stress jump \( \Delta \sigma \) across the shock front and a measure of the strain rate within the shock layer \( \dot{\varepsilon}_{SG} \) defined as the maximum time derivative of the particle velocity divided by the steady-wave shock velocity. From the analysis of experimental data, they found that for the materials considered, \( \dot{\varepsilon}_{SG} \) could be related to the stress jump \( \Delta \sigma \) by an empirical power law \( \dot{\varepsilon}_{SG} = \beta (\Delta \sigma)^{h_{SG}} \) with \( h_{SG} = 4 \). In their modeling of shock wave experiments, the exponent \( h_{SG} \) is related to the strain rate sensitivity \( m \) by the relationship \( h_{SG} = 2m \). From these results, it can be concluded that all materials tested exhibit almost the same strain rate sensitivity \( m = 0.5 \) for the deformation (strain rate) regime encountered in the shock experiments considered.

In this article, the problem of the propagation of steady plastic shock waves is revisited. The goal is to provide a further understanding of the relationship between steady plastic shocks, material parameters (strain rate sensitivity, etc.) and loading conditions. In particular, the role of viscous effects on the shock structure will be analyzed in detail. The general problem of compressive plane shock waves in metallic materials is considered. For illustration purposes, aluminum is chosen as the reference material. The constitutive framework used in this paper is similar to the one specified by Clifton\textsuperscript{17} in his analysis of transient elastoplastic waves in aluminum. The problem is formulated in a Lagrangian setting suitable for large deformations. The material response is characterized by a quasistatic tensile (or compression) test which provides in particular the strain hardening law in isothermal conditions. In addition the elastic response is characterized by ultrasonic measurements, up to second order elastic constants. Following Clifton,\textsuperscript{17} it is assumed that the deformation process is quasistentropic, for shocks of moderate amplitudes, and the temperature effects are neglected. Thermomechanical coupling which could have important effects for very strong shocks, can be easily included by keeping the essential structure of the proposed solution. Working within this simplified framework, an analytical solution for steady plastic waves is proposed. This approach seems to be sufficient to get accurate results for shocks of moderate strength (up to 10 GPa for aluminum). The model is validated by comparing the shock velocity, the particle velocity and the shock profiles with available experimental data. It should be emphasized that these results are obtained without using the Hugoniot data. In fact, it is shown that essential features of the Hugoniot diagram for moderate shock amplitudes can be directly retrieved from the present analysis. Surprisingly, the results remain fairly reasonable even for shock amplitudes much larger than 10 GPa.

Most of the solution is given in analytical form and hence provides a clear insight in to the problem. The shock structure is obtained by quadrature of a first order differential equation, made explicit under certain simplifying assumptions, and leading therefore to a closed form solution for steady plastic shocks. The effects of material parameters and loading conditions on the shock kinetics and shock structure are discussed. The shock width is also characterized in explicit form. Assuming an overstress formulation for the visco–plastic flow law (Perzyna),\textsuperscript{18} the effects of the strain rate sensitivity and of material hardening on the shock structure are analyzed. The relationship between the stress jump and the intensity of plastic strain rate within the shock layer is established. As in the work of Swegle and Grady,\textsuperscript{16} this relationship is shown to be well described by a power law. The exponent of this power law depends on the strain rate sensitivity and this dependence is formulated in explicit form. The results are compared to those of Swegle and Grady.\textsuperscript{16} Finally, a methodology for obtaining information on the viscoplastic response of materials subjected to very high strain rates using the shock waves experiments is discussed.

II. KINEMATIC AND CONSTITUTIVE FRAMEWORK

Consider a plane shock wave moving in the positive \( x \) direction. The displacement of a material particle has a component \( u \) in the \( x \) direction, the components being zero along the \( y \) and \( z \) directions of the orthonormal frame \( Oxyz \) (with unit vector basis \( e_1, e_2, \) and \( e_3 \), corresponding to uniaxial strain. The problem is formulated in a Lagrangian setting appropriate for large deformation. The kinematical and constitutive framework proposed by Clifton\textsuperscript{17} in his analysis of transient elastic-plastic waves is closely followed. The wave amplitudes considered were of moderate intensity and the process was treated as quasistentropic.

A. Kinematics

\( X \) denotes the Lagrangian coordinate of a particle in the initial state, \( x \) the current position, and by \( u \) the longitudinal displacement: \( x = X + u(X) \). The deformation gradient has the form

\[
\mathbf{F} = \lambda_1 e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3
\]

(1)

where the longitudinal stretch is defined by

\[
\lambda_1 = \frac{\partial X}{\partial x} = 1 + \frac{\partial u}{\partial X} = \frac{V}{V_0};
\]

where \( V \) and \( V_0 \) are, respectively, the current and the reference volumes of a material element. The deformation gradient can be decomposed in to elastic (\( \varepsilon \)) and plastic (\( \phi \)) parts through the standard multiplicative decomposition

\[
\mathbf{F} = \mathbf{F}^\varepsilon \mathbf{F}^\phi
\]

(2)

with

\[
\mathbf{F}^\varepsilon = \lambda_1^\varepsilon e_1 \otimes e_1 + \lambda_2^\varepsilon e_2 \otimes e_2 + \lambda_3^\varepsilon e_3 \otimes e_3,
\]

(3)

\[
\mathbf{F}^\phi = \lambda_1^\phi e_1 \otimes e_1 + \lambda_2^\phi e_2 \otimes e_2 + \lambda_3^\phi e_3 \otimes e_3.
\]

(4)

The axes \( y \) and \( z \) play equivalent roles, therefore the stretches in the \( y \) and \( z \) directions are identical as formulated in Eq.
(3). From Eq. (2), the total stretches are the products of elastic and plastic stretches; using in addition Eq. (1), it follows that
\[ \lambda_1 = \lambda_1^p \lambda_1^s, \quad \lambda_2 = \lambda_3 = \lambda_2^p \lambda_2^s = 1. \]  
(5)

Using plastic incompressibility
\[ \det \mathbf{F}^p = 1 \]
(6)

one obtains
\[ \lambda_1^s = \lambda_1 / \lambda_1^p, \quad \lambda_2^s = (\lambda_2^p)^{1/2}, \quad \lambda_2^p = (\lambda_1^s)^{-1/2}. \]  
(7)

and all the stretches can then be expressed in terms of the total (\( \lambda_1 \)) and plastic (\( \lambda_1^p \)) stretches in the direction of wave propagation.

**B. Constitutive equations**

In this section, the basic governing equations of the problem are introduced. The Lagrangian form of the equation of conservation of linear momentum is
\[ \frac{\partial T_1}{\partial X} = \rho_0 \frac{\partial \mathbf{v}}{\partial t}, \]
(8)

where \( t \) is time, \( \mathbf{v} = \partial u(X,t)/\partial t \) is the particle velocity, \( \rho_0 \) is the mass density in the reference configuration, and \( T_1 \) is the component of the Piola–Kirchhoff stress tensor in the wave propagation direction (force per unit area in the reference configuration). Using axisymmetry, the stress tensor can be expressed in the form
\[ \mathbf{T} = T_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + T_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + T_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \]
(9)

Note that \( T_1 \) coincides with the component \( \sigma_1 \) of the Cauchy stress tensor, because a material surface element normal to the propagation direction remains unchanged during the process. However, the transverse stress component \( T_2 \) is different from \( \sigma_2 \).

The kinematic compatibility equation has the form
\[ \frac{\partial \mathbf{v}}{\partial X} = \frac{\partial \mathbf{\lambda}_1}{\partial t}. \]
(10)

The conditions where the temperature variation can be neglected are considered here. Therefore, the equation of conservation of energy is not needed here.

The elastic constitutive law for an isentropic process can be written as
\[ T_1 = (\rho_0 \lambda_1^p) F_1(e_1^1, e_2^1), \]
(11)
\[ T_2 = \rho_0 (\lambda_2^p)^{1/2} F_2(e_1^2, e_2^2). \]
(12)

The elastic components of the strain tensor are given by
\[ e_1^1 = \lambda_2^s - 1 = (\lambda_1 / \lambda_1^s) - 1, \]
(13)
\[ e_2^2 = \lambda_2^s - 1 = (\lambda_2^p)^{1/2} - 1. \]
(14)

The functions \( F_1 \) and \( F_2 \) specified in Appendix A, are second order polynomials with respect to elastic deformations. The second order thermoelastic constants are traditionally obtained using ultrasonics measurements. In view of the relationships Eqs. (13) and (14), the stress components are solely dependent upon the stretches \( \lambda_1 \) and \( \lambda_1^p \).

The viscoplastic flow rule has the following functional form [see Eq. (38) in Ref. 17]
\[ \dot{\lambda}_1^p = - \frac{2}{3} \Phi(\tau, \gamma^p). \]
(15)

\( \tau \) is the maximum shear stress, which in the current configuration is given by
\[ \tau = 1/2(T_1 - T_2 / \lambda_1). \]
(16)

The hardening parameter \( \gamma^p \) is a measure of the plastic shear strain defined as
\[ \gamma^p = \int \Phi dt = - 3/2 \ln(\lambda_1^p). \]
(17)

Note that \( \Phi \) is in general temperature dependent, but the temperature effect is neglected here. The form adopted for \( \Phi \) is as follows:
\[ \Phi = b N \tilde{v}, \]
(18)

where \( b \) is a constant (in a single crystal \( b \) can be identified with the length of the Burgers vector associated with single slip); \( N = N(\gamma^p) \) is the mobile dislocation density, a function of the cumulated plastic strain. Following Johnson and Barker,11 the following form is adopted:
\[ N = N_{m0}[1 + (\alpha_{b0} \gamma^p / b N_{m0})] \exp(- \alpha_{l0} \gamma^p), \]
(19)

where \( N_{m0} \) is the initial mobile dislocation density, \( N_{l0} \) is the initial total dislocation density, \( \alpha_{b0} \) is the breeding coefficient, and \( \alpha_{l0} \) is the trapping coefficient. Values of these parameters are specified for aluminum in Table I. The average dislocation velocity \( \tilde{v} = \tilde{v}(\tau, \gamma^p, \theta) \) is in general dependent upon the resolved stress \( \tau \), the accumulated plastic strain \( \gamma^p \), and the temperature \( \theta \). The temperature dependence shall not be considered here.

The stress dependence of \( \tilde{v} \) is assumed to be described by an overstress model. In this theory, viscoplastic flow occurs when the resolved stress \( \tau \) is exterior to the elastic domain \((|\tau| > \tau_a)\) and vanishes otherwise \((|\tau| \leq \tau_a)\), with \( \tau_a \) being the yield stress in pure shear. Strain hardening is described in terms of \( \gamma^p \) by the following power law:
\[ \tau_a = \tau_{a0} \left[ 1 + \left( \frac{\gamma^p}{\gamma_{00}^p} \right)^{1/n} \right], \]
(20)

where \( \tau_{a0}, \gamma_{00}^p, \) and \( n \) are material parameters that can be obtained from quasistatic tests (see Table I). Ideally, it would be better to use data obtained from adiabatic tests, since these data will include thermal softening effects due to the self-heating produced by plastic dissipation. However, this modification would just slightly influence the results for plastic shock waves of moderate amplitude. The reason is that for small strains the heat produced by dissipation is small enough that can be neglected. For larger strains, \( \tau_a \) could be affected by thermal softening, but then the compressive stress \( \sigma \) becomes large with respect to \( \tau_a \) and knowing the precise value of \( \tau_a \) is of not significant importance.

Under compressive loading \( \tau \) is negative, and the yield condition is
\[ \tau + \tau_a = 0. \]
(21)
Viscoplastic flow occurs when
\[ \tau + \tau^* < 0. \]  
(22)

It is assumed that \( \tilde{\nu} \) is a power law function of the overstress
\[ \tilde{\nu} = c_1 \left( \frac{\tau + \tau^*}{T_1^*} \right)^M, \]  
(23)

where \( c_1 \) and \( T_1^* \) are material constants, and \( M = 1/m \) is the inverse of the strain rate sensitivity \( m \). The following values will be used \( c_1 = 0.168 \text{ ms}^{-1} \), \( M = 1.78 \), and \( T_1^* = 1.6 \text{ MPa} \). Clifton\(^{17}\) and Johnson and Barker\(^{11}\) have used a slightly different form
\[ \tilde{\nu} = c_1 \left( \frac{\tau + \tau^*}{T^*} \right)^M, \]  
(24)

with \( M_c = 2.14 \), \( c_s = 3197 \text{ m/s} \), and \( T^* = 203 \text{ MPa} \). Note that Eq. (23) would coincide with Eq. (24) for \( M = M_c \) and \( T_1^* = 2.03 \text{ MPa} \).

The choice of the parameters in Table I will be justified later by considering the shock profiles recorded in plate impact experiments. The methodology used to identify these parameters will be discussed in Sec. VI.

Finally the problem of the propagation of steady plastic waves is governed by a system of five equations [Eqs. (8), (10), (11), (12), and (15)], the variables being \( \lambda_1, \lambda_1^p, \nu, T_1, \) and \( T_2 \). Furthermore, if \( T_1, T_2, \) and \( \tau \) are expressed in terms of \( \lambda_1 \) and \( \lambda_1^p \) with the use of Eqs. (13), (14), and (16), one obtains, after substitution of these expressions into Eqs. (8) and (15), a system of three equations [Eqs. (8), (10), and (15)] for the unknowns \( \lambda_1, \lambda_1^p, \) and \( \nu \).

The values of the material parameters used in this article for 6061-T6 aluminum are given in Table I; for further information on thermo–elastic properties see Clifton.\(^{17}\) During the parametric analysis in Sec. VI, some values of the material parameters of Table I will be varied for illustrative purposes.

### C. Isothermal compression curve

In Fig. 1(a), the variation of the compressive stress \( \sigma = -T_1 \) for 6061-T6 aluminum is reported in terms of the total stretch \( \lambda_1 \) for a slow compression process; then viscous effects can be neglected and the stress state remains on the yield surface. In Fig. 1(a), the stretch \( \lambda_1 \) is decreased from the initial value \( \lambda_1 = 1 \) to \( \lambda_1 = 0.9 \). The elastic limit is reached at \( \lambda_1^\sigma = 0.995 \), for the value \( \sigma^\sigma = 0.473 \text{ GPa} \) of the compressive stress. For \( \lambda_1^\sigma = \lambda_1 \leq 1 \), the response is purely elastic and the plastic stretch is equal to \( \lambda_1^p = 1 \). When the material is further compressed, one enters the elasto–plastic regime \( \lambda_1 < \lambda_1^\sigma \). The resulting stress–stretch response is the slightly convex curve shown in Fig. 1(a) corresponding to states \( (\lambda_1, \lambda_1^p) \) for which the yield criterion Eq. (21) is satisfied. More precisely, for a given \( \lambda_1 \), the plastic stretch \( \lambda_1^p \) is determined in terms of \( \lambda_1 \) so as to satisfy the yield criterion \( \tau (\lambda_1, \lambda_1^p) + \tau_\sigma (\lambda_1^p) = 0 \) (cf. Table I). This relationship defines a correspondence between \( \lambda_1 \) and \( \lambda_1^p \) which can be denoted as
\[ \lambda_1^p = f_\nu (\lambda_1), \quad \lambda_1 = g_\nu (\lambda_1^p). \]  
(25)

Therefore, the compression curve can be parameterized by \( \lambda_1, \sigma = -T_1 (\lambda_1, f_\nu (\lambda_1)) \). \( C_\nu \) denotes the elasto–plastic part of the compression diagram. The subscript \( (\cdot)_\nu \) is used in Eq. (25) to indicate that these relationships are only valid on \( C_\nu \).

It is worth noting that the resolved stress \( \tau \) vanishes if \( \lambda_1^p = \lambda_1^{23/3} \). For stress levels large enough so that the threshold stress \( \tau_\sigma \) can be neglected, the yield criterion can be written

<table>
<thead>
<tr>
<th>TABLE I. Material parameters for 6061-T6 aluminum at room temperature (25 °C) and atmospheric pressure.</th>
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<tbody>
<tr>
<td>( \rho = 2703 \text{ kg/m}^3 )</td>
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<tr>
<td>( c_L = 6368 \text{ m/s} )</td>
</tr>
<tr>
<td>( c_S = 3197 \text{ m/s} )</td>
</tr>
<tr>
<td>Elastic Constants</td>
</tr>
<tr>
<td>( a_2 = c_L^2/2 = 20.28 \times 10^4 \text{ (m/s)}^2 ); ( a_3 = -2c_S^2 = -20.44 \times 10^4 \text{ (m/s)}^2 ); ( a_4 = -66.4 \times 10^4 \text{ (m/s)}^2 ); ( a_5 = 157.5 \times 10^4 \text{ (m/s)}^2 ); ( a_6 = -142.8 \times 10^4 \text{ (m/s)}^2 )</td>
</tr>
<tr>
<td>Plastic characteristics provided by a quasistatic tensile test (hardening law [Eq. (20)])</td>
</tr>
<tr>
<td>( \tau_0 = 120 \text{ MPa} )</td>
</tr>
<tr>
<td>( \tilde{\gamma}_0 = 0.52 )</td>
</tr>
<tr>
<td>( n = 1.55 )</td>
</tr>
<tr>
<td>Conjectured viscoplastic properties (see [Eqs. (19) and (23)])</td>
</tr>
<tr>
<td>( M = 1/m = 1.78 )</td>
</tr>
<tr>
<td>( c_1 = 0.168 \text{ ms}^{-1} )</td>
</tr>
<tr>
<td>( T_1^* = 1.6 \text{ MPa} )</td>
</tr>
<tr>
<td>( b = 0.286 \times 10^{-9} \text{ m} )</td>
</tr>
<tr>
<td>( N_{ml} = 0.818 \times 10^{15} \text{ m}^{-2} )</td>
</tr>
<tr>
<td>( N_o = 0.818 \times 10^{14} \text{ m}^{-2} )</td>
</tr>
<tr>
<td>( \alpha^{cl} = 3.5 \times 10^{5} \text{ m}^{-1} )</td>
</tr>
<tr>
<td>( \alpha = 0 )</td>
</tr>
<tr>
<td>Thermomechanical parameters</td>
</tr>
<tr>
<td>( b_1 = -593 \text{ K} ); ( b_2 = -130 \text{ K} ); ( b_3 = 1350 \text{ K} )</td>
</tr>
<tr>
<td>( c_p = 908 \text{ J/(K g)}^{-1} \text{(K)}^{-1} )</td>
</tr>
<tr>
<td>( \beta = 0.9 )</td>
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<td>( * )</td>
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</table>
III. STEADY STATE SHOCK WAVES

For shocks with large enough amplitude, steady profiles can be achieved. These steady shock profiles propagate without change in form, which results from the balance between the steepening effect of the nonlinear (viscoplastic) material response and the smoothing effect of dissipation. For a certain range of shock amplitudes, two compressive shock waves are generated. The first shock is the elastic precursor, which compresses the material to the elastic limit. The state reached at the rear of the elastic precursor will be referred to as state (+). The corresponding compressive stress, particle velocity, and stretches are, respectively, denoted by \( \sigma^+, v^+, \lambda^+, \lambda^{pe+} \). The elastic precursor is followed by a plastic wave, which further compresses the material to the final stress level \( \sigma^- = -T_1^- \). The transition from \( \sigma^+ \) to \( \sigma^- \) occurs within the shock front, which is a thin layer where the material is rapidly deformed at very high strain rates. While in the present model, material viscosity does not affect the shape of the compression curve \( C_Y \), viscous effects play a key role in structuring the shock front layer. If viscous effects are neglected, this layer can be idealized as being infinitely thin and the stress sustains an instantaneous jump \( \Delta \sigma = \sigma^- - \sigma^+ \) at the shock front.

A. Basic equations

To model the steady structure of plastic waves, consider an observer moving with the Lagrangian velocity \( C \) of the plastic shock with respect to the reference configuration. This velocity is constant for a steady state shock. For this observer, the shock profile has a steady structure and all quantities can be expressed in terms of a moving coordinate \( \xi = X - Ct \).

Therefore, Eqs. (8) and (10) can be written as

\[
\frac{dT_1}{d\xi} = -C \rho_0 \frac{dv}{d\xi} \quad (28)
\]

\[
\frac{dv}{d\xi} = -C \frac{d\lambda_1}{d\xi} \quad (29)
\]

The analysis is conducted by considering that the local frame is stretched in the \( \xi \) direction so that states (+) and (−) are, respectively, at \( \xi = +\infty \) and \( \xi = -\infty \). Upon integration with respect to \( \xi \), it follows, for any point within the plastic shock layer

\[
T_1 - T_1^+ = -\rho_0 C (v - v^+), \quad (30)
\]

\[
v - v^+ = -C (\lambda_1 - \lambda_1^+). \quad (31)
\]

From Eqs. (30) and (31), it is established that the stress \( T_1 \) is linearly related to the stretch \( \lambda_1 \),

\[
T_1 - T_1^+ = \rho_0 C^2 (\lambda_1 - \lambda_1^+). \quad (32)
\]

This relationship defines the so called Rayleigh line, a straight line in the \((T_1, \lambda_1)\) diagram. Considering in particular the state (−), one has

\[
T_1 - T_1^- = -\rho_0 C (v^- - v^+), \quad (33)
\]

\[
v^- - v^+ = -C (\lambda_1^- - \lambda_1^+), \quad (34)
\]

\[
T_1^- - T_1^- = \rho_0 C^2 (\lambda_1^- - \lambda_1^+). \quad (35)
\]
The state (+) corresponds to initial yielding, for which plastic deformation is still nonexistent. Thus, at state (+), one has \( \lambda^+ = 1 \), and the yield condition Eq. (21) takes the form

\[
1/2(T_1(\lambda^+,1) - T_2(\lambda^+,1)/\lambda^+\gamma + \tau_{\alpha 0} = 0.
\]

The stretch \( \lambda^+ \) is determined as the solution to Eq. (36). The corresponding stress is given by

\[
T^+ = T_1(\lambda^+,1). \quad (37)
\]

For 6061-T6 aluminum, whose material characteristics are defined in Table I, \( \lambda^+ = 0.99578 \) and \( \sigma^+ = -T^+ = 473 \text{ MPa} \). The particle velocity \( \nu^+ \) is given by a relationship similar to Eq. (34), where \( C \) is replaced by the velocity of the elastic precursor

\[
c\alpha = \sqrt{\frac{\sigma^+}{\rho_0(1 - \lambda^+)}}.
\]

(38)

The resulting follows are obtained for the elastic precursor wave speed and particle velocity:

\[
c\alpha = 6434 \text{ m/s}
\]

and

\[
\nu^+ = 27.2 \text{ m/s}.
\]

State (−) is entirely determined by a single input parameter characterizing the shock amplitude. This parameter can either be the stress amplitude \( \sigma^* = -T^\gamma \) or the stretch \( \lambda^\gamma \) or the shock velocity \( C \). Assuming for example that \( \lambda^\gamma \) is given, the plastic stretch \( \lambda^P \) can be calculated by considering the yield condition Eq. (21) at state (−)

\[
1/2[T_1(\lambda^\gamma,1) - T_2(\lambda^\gamma,1)/\lambda^\gamma\gamma + \tau_{\alpha 0} = 0 \quad (39)
\]

with

\[
\tau_{\alpha 0} = \tau_{\alpha 0}\left[1 + \left(-\frac{3}{2}\frac{\ln(\lambda^\gamma)}{\gamma_0}\right)^{1/2} \right].
\]

The stress is given in terms of stretches by

\[
\sigma^* = -T^\gamma = T_1(\lambda^\gamma,1,\lambda^P).
\]

Shocks velocity \( C \) and particle velocity \( \nu^\gamma \) follow from Eqs. (35) and (34). Similar calculations can be made if, in place of \( \lambda^\gamma \), the input parameter is the stress \( T^\gamma \).

B. Shock structure

Once the states (+) and (−) are characterized, the shock structure is entirely determined from the flow equation (15), which, by considering Eq. (27), can be expressed as

\[
\frac{d\lambda^P}{d\xi} = \frac{2\lambda^P}{3C} \Phi(\tau, \gamma^P)
\]

with

\[
\gamma^P = -3/2 \ln(\lambda^P).
\]

(42)

In addition, all states within the plastic shock are constrained to be on the Rayleigh line Eq. (32). This constraint provides an equation for determining the plastic stretch \( \lambda^P \) in terms of the total stretch \( \lambda^\gamma \) in the shock layer

\[
T_1(\lambda^\gamma,\lambda^P) - T_1 = \rho_0C^2(\lambda^\gamma - \lambda^P).
\]

Let us denote this functional dependence by

\[
\lambda^P = g_\rho(\lambda^\gamma).
\]

With the aid of Eqs. (11), (12), (16), and (44), the stress \( \tau \) can be expressed uniquely in terms of \( \lambda^\gamma \). Finally, considering that \( \gamma^P \) is related to \( \lambda^P \) by Eq. (17), \( \Phi \) appears to be solely the function of \( \lambda^P \),

\[
\Phi(\lambda^P) = \Phi(\tau(\lambda^\gamma), \gamma^P(\lambda^\gamma)).
\]

(45)

The subscript \( R \) used in Eqs. (44) and (45) indicates that these relationships are only valid within the shock structure (i.e., on the Rayleigh line). Finally, with Eq. (45), the flow rule Eq. (42) can be written in the form of a first order differential equation with respect to the single unknown \( \lambda^P \)

\[
\frac{d\lambda^P}{d\xi} = \frac{2}{3C}\lambda^P \Phi(\lambda^P).
\]

(46)

Note that the viscoplastic flow vanishes (\( \Phi = 0 \)) at states (+) and (−), i.e., at \( \xi = \pm \infty \) and \( \xi = \mp \infty \), respectively, since the yield criterion Eq. (21) is identically satisfied at these states. The flow equation (46) determines the shock structure.

Within the shock layer, the stretches \( \lambda^\gamma \) and \( \lambda^P \) are related by Eq. (44). A remarkable feature of the present model is that this relationship can be written in an explicit form. As shown in Appendix A, the function \( F_1(\epsilon^1_1, \epsilon^2_2) \), involved in the expression Eq. (11) for \( T_1 \) is a second order polynomial with respect to the arguments \( \epsilon^{1}_1 \) and \( \epsilon^{2}_2 \)

\[
F_1(\epsilon^{1}_1, \epsilon^{2}_2) = A_1\epsilon^{1}_1 + A_2\epsilon^{1}_1 + B_1\epsilon^{2}_2 + B_2\epsilon^{2}_2 + C\epsilon^{1}_1\epsilon^{2}_2 + D\epsilon^{1}_2\epsilon^{2}_2,
\]

with \( A_1, A_2, B_1, B_2, D \) given by Eq. (A5). Since \( \lambda^P = 1 + \epsilon^{2}_2 \), the relationship Eq. (32) describing the Rayleigh line provides a second order algebraic equation with respect to \( \epsilon^{1}_1 \)

\[
A_1\epsilon^{2}_1^2 + B\epsilon^{1}_1 + G = 0
\]

with

\[
B = A_1 + D\epsilon^{2}_2 - C^2(1 + \epsilon^{2}_2)^4,
\]

\[
G = B_1\epsilon^{2}_2 + B_2\epsilon^{2}_2 + (C^2\lambda^\gamma - T_1^\gamma + \rho_0)(1 + \epsilon^{2}_2)^2 - C^2(1 + \epsilon^{2}_2)^4,
\]

where \( C \) is the velocity of the plastic shock wave. Considering that \( \epsilon^{2}_2 = 0 \) at the state (+), the only root of Eq. (48) having a physical meaning is

\[
\epsilon^{1}_1 = -B + \sqrt{B^2 - 4A_2G}/2A_2
\]

(50)

From Eq. (50), \( \epsilon^{1}_1 \) can be expressed in terms of \( \epsilon^{2}_2 \) or \( \lambda^P \) by

\[
\epsilon^{1}_1 = \frac{f(\epsilon^{2}_2) - f(\lambda^P)^{12} - 1}{f(\lambda^P)^{12} - 1},
\]

(51)

where the function \( f \) has the form
Finally, the explicit form of Eq. (44) is obtained by using Eqs. (13) and (51)

$$\lambda_1 = g_R(\lambda_1^p) = \lambda_1^p \{ 1 + f((\lambda_1^p)^{1/2} - 1) \}. \quad (53)$$

Note that the longitudinal strain rate \( \dot{\varepsilon}_p \) can be written in terms of \( \lambda_1^p \)

$$\dot{\varepsilon}_p = \frac{\dot{\lambda}_1}{\lambda_1} = -\frac{2}{3} \lambda_1^p \frac{g_R(\lambda_1^p)}{g(\lambda_1^p)} \Phi_R(\lambda_1^p). \quad (54)$$

Using Eqs. (53), (54), and (3), the total strain rate \( \dot{\varepsilon} \) can also be expressed in terms of \( \lambda_1^p \)

$$\dot{\varepsilon} = -\frac{2}{3} \lambda_1^p \frac{g_R(\lambda_1^p)}{g(\lambda_1^p)} \Phi_R(\lambda_1^p). \quad (55)$$

The explicit form of the derivative \( g_R \) with respect to \( \lambda_1^p \), is given in Appendix B. Integration of the differential equation (46) provides the shock structure, i.e., the variation of \( \lambda_1^p \) with respect to the position \( \xi \). The evolution of the material state with \( \xi \) follows, because the material state is characterized by \( (\lambda_1, \lambda_1^p) \), and \( \lambda_1 \) is related to \( \lambda_1^p \) by Eq. (53).

**IV. COMPARISON WITH EXPERIMENTS**

In this section, the model predictions are compared with experimental data for aluminum.\(^{1,11}\)

**A. Shock velocity and particle velocity**

In Fig. 2, the isothermal compression curve for a shock of stress amplitude \( \sigma^* = 3.7 \) GPa is considered. Points \( J \) and \( K \) on the compression curve \( C_Y \) correspond, respectively, to the states (+) and (−), ahead and at the rear of the plastic shock. According to Eq. (35), the slope of the segment \( JK \) is \( -\rho_1 C^2 \). As shown before, the stretches \( \lambda_1 \) and \( \lambda_1^p \) at \( K \) are determined so as to satisfy the yield condition Eq. (21) at the given stress \( \sigma^- \). The plastic shock velocity \( C \) is obtained from Eq. (35) and the particle velocity \( \nu^- \) from Eqs. (33) or (34). The resulting values are: \( \lambda_1 = 0.9577, \lambda_1^p = 0.9745, C = 5602 \) m/s, and \( \nu^- = 240 \) m/s. Results for \( \sigma^* = 15 \) GPa are shown in Fig. 2(b), and Table II. Other results obtained for a variety of stress amplitudes are compared in Table II to experimental data for aluminum reported by Rice et al.\(^{1}\) and Johnson and Barker.\(^{11}\)

This comparison shows that for moderate shocks with stress amplitude less than 20 GPa, the present simplified constitutive framework provides quite good information concerning the shock kinematics. Even for much stronger shocks up to 100 GPa, the model predictions remain acceptable. The fact that dissipation and temperature effects can be neglected for moderate shocks is illustrated in Fig. 2. The energy dissipated by viscous effects is proportional to the area between the Rayleigh line and the compression curve \( C_Y \). In addition, one has to consider the energy dissipated by plastic work. For \( \sigma^- < 20 \) GPa, this dissipated energy is not large enough to influence significantly the compressive stress. The temperature rise at the rear of the plastic shock is evaluated in Appendix C.

In Fig. 3, the variation of the plastic shock velocity \( C \) is given in terms of the particle velocity \( \nu^- \) at the rear of the plastic shock. The range of velocities corresponds to stress amplitudes from 2.1 to 9 GPa (as in Johnson and Barker experiments). The linear dependence is in agreement with experiments and the predicted slope \( S = 1.347 \) is close to the value \( S = 1.337 \) obtained by Swegle and Grady\(^{16}\) from the experimental data of Johnson and Barker.\(^{11}\) The Eulerian value of the shock velocity is given by, (see for instance Swegle and Grady\(^{16}\))

$$C_{\mathrm{eul}} = \nu^+ + c^+ + S(\nu^- - \nu^+), \quad (56)$$

where \( c^+ \) is the sound velocity associated with the state (+). When the stress amplitude \( \sigma^- \) is decreased to the value \( \sigma^+ \), it follows that \( \nu^- \rightarrow \nu^+ \) and \( C_{\mathrm{eul}} \rightarrow c^+ \). Therefore the Lagrangian shock velocity is tending to the value \( C = c^+ \). Considering the results reported in Fig. 3, one obtains in this...
TABLE II. Experimental data for aluminum (Johnson and Barkera and Rice, McQueen and Walshb) are compared with the present model. $\sigma^-$, $\nu^-$, and $\lambda_i^- = V^i/V_0$ are respectively, the stress, the particle velocity, and the stretch at the rear (steady state) of the plastic shock front.

<table>
<thead>
<tr>
<th>Stress amplitude $\sigma^-$ (GPa)</th>
<th>Plastic shock velocity $C_n$ (m/s)</th>
<th>Particle velocity $\nu^-$ (m/s)</th>
<th>Stretch $\lambda_i^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a 2.1</td>
<td>...</td>
<td>5458</td>
<td>130</td>
</tr>
<tr>
<td>a 3.7</td>
<td>...</td>
<td>5602</td>
<td>240</td>
</tr>
<tr>
<td>a 9</td>
<td>...</td>
<td>6017</td>
<td>540</td>
</tr>
<tr>
<td>b 10</td>
<td>6125</td>
<td>6088</td>
<td>580</td>
</tr>
<tr>
<td>b 15</td>
<td>6475</td>
<td>6414</td>
<td>831</td>
</tr>
<tr>
<td>b 20</td>
<td>6793</td>
<td>6701</td>
<td>1057</td>
</tr>
<tr>
<td>b 30</td>
<td>7350</td>
<td>7204</td>
<td>1465</td>
</tr>
<tr>
<td>b 100</td>
<td>10 126</td>
<td>9573</td>
<td>3546</td>
</tr>
</tbody>
</table>

a See Ref. 1. 
b See Ref. 1.

limiting case, $C = c^+ = 5320$ m/s for $\nu^- = \nu^+ = 27$ m/s. This result compares well with $c_0 = c^+ + \nu^+ = 5380$ m/s, i.e., $c^+ = 5353$ m/s from Swegle and Grady.16

It is worth emphasizing that the results obtained in this section are independent of the viscous part of the material response specified in Eqs. (15) and (18). Results depend solely on the shape of the compression curve $C_\gamma$. This curve is governed by the elastic law Eqs. (11), (12) and the plastic hardening law Eq. (20). Material characteristics are provided by a single quasistatic elastic–plastic tensile test and by ultrasonic measurements to estimate the second order elastic constants.

B. Shock structure

The shock structure is obtained by integration of the differential equation (46). Results shown in Fig. 4 are for $\sigma^- = 3.7$ GPa and values of viscous parameters ($M = 1.78$, $T^p_n = 1.6$ MPa) as listed in Table I. The variation of $\lambda_1$, $\lambda_1^p$ and of the compression stress $\sigma$ with respect to $\xi$, are displayed in Figs. 4(a) and 4(b). Figure 4(c) shows the evolution of the total and the plastic strain rates, respectively, $\dot{\varepsilon}$ and $\dot{\varepsilon}^p$ (absolute values are shown). It appears that max $|\dot{\varepsilon}|$ is about 40% larger than max $|\dot{\varepsilon}^p|$. Figure 4(d) shows the particle velocity $\nu^-$ at the rear of the plastic front in terms of time. Such profiles have been recorded for aluminum by Johnson and Barker,11 at different shock strengths ($\sigma^- = 2.1$ GPa, $\sigma^- = 3.7$ GPa, and $\sigma^- = 9$ GPa). It must be noted that for the largest stress level, $\sigma^- = 9$ GPa, the particle velocity measurements were made at the limit of the time resolution of the experimental setup available at that time.

In Fig. 5, the shock profiles (particle velocity $\nu^-$ versus time) are displayed for different values of the stress amplitude $\sigma^-$. Model predictions are compared to the experimental data of Johnson and Barker.11 The good agreement that is evident is due to the appropriate choice of the parameters ($M$, $T^p_n$) controlling the viscous flow. A method to determine these parameters from the shock wave experimental data will be discussed in Sec. VI.

An interesting feature of the model is to provide an estimate of the shock width $w_1$. Considering Eq. (46), the following estimate is obtained:

$$w_1 = \frac{1 - \lambda_1^p}{\max(d\lambda_i^p/d\xi)} \cdot \frac{3C}{2 \max[\lambda_i^\sigma \Phi_R(\lambda_i^\sigma) \max(\lambda_i^p)]}.$$  \hspace{1cm} (57)

This expression is the ratio of the variation of the plastic stretch $(1 - \lambda_i^p)$ across the shock front, by a measure of the variation rate which is taken here as $\max(d\lambda_i^p/d\xi)$. To a very good approximation, the maximum of $\lambda_i^\sigma \Phi_R(\lambda_i^\sigma)$ occurs at the middle of the shock layer. This is verified by Fig. 6, where the variations of $\lambda_i^\sigma \Phi_R(\lambda_i^\sigma)$ and $\Phi(\lambda_i^\sigma)$ are shown in terms of $\lambda_i^\sigma$ for the stress amplitude $\sigma^- = 3.7$ GPa. Furthermore, at the middle of the shock layer, the plastic stretch $\lambda_i^p$ has nearly reached half of its variation and can be approximated by

$$\bar{\lambda}_i^p = (1/2)(1 + \lambda_i^p).$$  \hspace{1cm} (58)
Another estimate for shock width can be found by considering the following form of Eq. (58):

\[ w_1 = \frac{3C}{2} \frac{1 - \lambda_1^{p^-}}{\Phi_B(\lambda_1^{p^-})}. \]  

(59)

Another estimate for shock width can be found by considering the following form of Eq. (46):

\[ \frac{d}{d\xi} \ln(\lambda_1^{p^+}) = \frac{2}{3C} \Phi(\tau, \gamma'), \]

(60)

from which one gets

\[ w_2 = \frac{3C}{2} \frac{\ln(1/\lambda_1^{p^-})}{\max \Phi_B(\lambda_1^{p^-})}. \]  

(61)

and the estimate

\[ w_2 = \frac{3C}{2} \ln(1/\lambda_1^{p^-}) \frac{1}{\Phi_B(\lambda_1^{p^-})}. \]  

(62)

For moderate deformations, \(1 - \lambda_1^{p^-}\) is a small quantity and the estimates Eqs. (59) and (62) provide very close results. For the loading, \(\sigma^- = 3.7\) GPa, one obtains

\[ w_1^g = w_2^g = 196 \text{ } \mu\text{m}; \quad w_1^e = 201 \text{ } \mu\text{m}. \]  

(63)

In Fig. 7, the evolution of the characteristic shock width \(w_1\) is reported in terms of the shock amplitude \(\sigma^-\). The value of \(w_1^g\) is also plotted but is almost indistinguishable from \(w_1\). Two choices of the viscous parameters have been considered: (i) \((M, T_1^g) = (1.78, 1.6\) MPa) as in Table I, and (ii) \((M, T_1^g) = (2.2, 3.33\) MPa). Note that \(w_1\) grows to very large values when the stress amplitude becomes smaller than 2 GPa [case (i) \(w_1 = 10\) cm for \(\sigma^- = 1\) GPa]. Actually, from the expression Eq. (59) for the characteristic width, it follows

\[ \frac{w_1^g}{w_1^e} = \frac{196}{201} = 0.9745. \]  

(64)

FIG. 5. Shock profiles showing the evolution of the particle velocity \(\nu^-\) with time, for different stress amplitudes, \(\sigma^- = 2.1\) GPa, \(\sigma^- = 3.7\) GPa, and \(\sigma^- = 9\) GPa. The model predictions are compared with experimental data for aluminum obtained by Johnson and Barker (see Ref. 11).

FIG. 6. Variation, within the shock layer, of the plastic strain rate \(\dot{\gamma}^p = \Phi (\text{continuous line})\) and of \(\lambda_1^{p^+}(\lambda_1^{p^-})\) (dashed line) in terms of the plastic stretch \(\lambda_1^{p^-}\). The shock stress amplitude is \(\sigma^- = 3.7\) GPa. Plastic flow vanishes at the states (+) and (−) corresponding, respectively, to \(\lambda_1^{p^-} = 1\) and \(\lambda_1^{p^-} = 0.9745\).

FIG. 7. Evolution of the characteristic shock width \(w_1\) with the stress amplitude \(\sigma^-\). Results for \(w_1^g\) are also reported on this figure, but they are hardly distinguishable from \(w_1\). Two choices of the viscous parameters have been considered: (i) \((M, T_1^g) = (1.78, 1.6\) MPa) as in Table I, and (ii) \((M, T_1^g) = (2.2, 3.33\) MPa). The following values of \((\sigma^-\) (GPa), \(w_1\) (\(\mu\)m)) are associated to the corresponding points (circles or dots): (i) \(A = (15.2, 80)\), \(B = (12.5, 0.09)\), \(C = (9.11.5)\), \(D = (6.39)\), \(E = (3.7, 196)\), \(F = (2.1, 168)\), \(G = (1.0 \times 10^6)\). (ii) \(A = (15.1, 36)\), \(B = (12.2, 7)\), \(C = (9.6, 8)\), \(D = (6.27)\), \(E = (3.7, 171)\), \(F = (2.1, 1964)\), \(G = (1.0 \times 10^6)\).
that \( w^{a}_{1} \to \infty \) when \( \sigma^{-} \to \sigma^{-} \) [since \( \lambda_{1}^{p} \to 1, \lambda_{1}^{p} \to 1, \Phi(\lambda_{1}^{p}) \to 0 \)]. According to Bland, a propagation distance of at least five times the shock width is needed to achieve steady state conditions at the shock front. For \( \sigma^{-} = 1 \) GPa this means that the specimen should be at least 50 cm thick! For \( \sigma^{-} = 3.7 \) GPa, the thickness should be larger than 1 mm.

In Fig. 8, the stress jump \( \Delta \sigma = \sigma^{-} - \sigma^{+} \) is given in terms of the quantities \( \max[\dot{\varepsilon}^{p}] \) and \( \max[\dot{\varepsilon}] \) evaluated within the shock layer (i.e., on the Rayleigh line), with \( \dot{\varepsilon}^{p} \) and \( \dot{\varepsilon} \) being, respectively, the longitudinal plastic strain rate Eq. (54) and the total strain rate Eq. (55). In the log–log plot of Fig. 8, two straight lines are obtained which are nearly parallel, suggesting the following laws:

\[
\begin{align*}
\max[\dot{\varepsilon}^{p}] &= \beta_{p}(\Delta \sigma)^{\rho_{p}}, \\
\max[\dot{\varepsilon}] &= \beta(\Delta \sigma)^{\rho}.
\end{align*}
\]

The slopes of the lines in Fig. 8 are \( 1/h_{p} \) and \( 1/h \). It will be shown later that \( h_{p} \) and \( h \) depend on \( M \) but are not dependent on \( T_{1}^{s} \), while \( \beta_{p} \) and \( \beta \) depend on both \( M \) and \( T_{1}^{s} \). For the values of material parameters considered in Table I, \( h = 3.92 \) and \( h_{p} = 3.98 \). It is worth noting that the exponent of \( \Delta \sigma \) from the present model agrees well with that identified by Swegle and Grady from the experimental results of Johnson and Barker, \( h_{SG} = 4 \). However, the rate sensitivity parameter is \( M = 1/m = 1.78 \) in the present model, while it was \( M = 2 \) in the approach developed by Swegle and Grady. Further discussion of this issue is provided in Sec. VI.

V. CLOSED FORM SOLUTIONS FOR THE SHOCK STRUCTURE

For certain values of the strain rate sensitivity \( m = 1/M \), the differential equation (46) has an explicit closed form solution. It was shown before that on the Rayleigh line \( \tau + \tau_{p} \) can be expressed in terms of the single variable \( \lambda_{1}^{p} \). Using the notations of Fig. 2(a), the Rayleigh line intersects the compression curve \( C_{Y} \) at points \( J \) and \( K \) where \( \lambda_{1}^{p} = 1 \) and \( \lambda_{1}^{p} = \lambda_{1}^{p} \). Since \( \tau + \tau_{p} = 0 \) at \( J \) and \( K \), one can approximate \( \tau + \tau_{p} \) on the Rayleigh line by a lineary dependence upon \( \lambda_{1}^{p} - 1 \) and \( \lambda_{1}^{p} - \lambda_{1}^{p} \), respectively, in the vicinity of \( J \) and \( K \). Taking into account the exponent \( M \) in expression (23) for the dislocation velocity, \( \Phi \) can be approximated by the following function of \( \lambda_{1}^{p} \):

\[
\Phi(\lambda_{1}^{p}) = \frac{4M\Phi_{\max}}{(1-\lambda_{1}^{p}M)(1-\lambda_{1}^{p}M)(\lambda_{1}^{p} - \lambda_{1}^{p} - \lambda_{1}^{p})^M}. \tag{65}
\]

The right hand side of Eq. (65) has been normalized to have the maximum \( \Phi_{\max} \) of \( \Phi \) at the mean value of the plastic stretch \( \lambda_{1}^{p} = (1/2)(1 + \lambda_{1}^{p} - \lambda_{1}^{p}) \), in accordance with results of Fig. 6. For a moderate deformation, the right hand side of Eq. (42) can be approximated by replacing \( \lambda_{1}^{p} \Phi \) by \( \lambda_{1}^{p} \Phi \)

\[
\frac{d\lambda_{1}^{p}}{d \xi} = K(1-\lambda_{1}^{p}M)(1-\lambda_{1}^{p}M), \tag{66}
\]

where

\[
K = \frac{2\lambda_{1}^{p} - 4M\Phi_{\max}}{3C(1-\lambda_{1}^{p}M)^2}. \tag{67}
\]

The solution for the differential equation (66) has the form:

\[
\xi = K^{-1}(1-\lambda_{1}^{p}M)^{-1}(1-\lambda_{1}^{p}M)^{-1}Z[(\lambda_{1}^{p} - \lambda_{1}^{p})/(1-\lambda_{1}^{p})], \tag{68}
\]

where \( Z \) is the function

\[
Z(x) = \int \frac{dx}{(1-x)^{M}x^{M}} \quad (0 \leq x \leq 1). \tag{69}
\]

When \( M \) is an integer, the integral Eq. (69) can be evaluated explicitly. For instance for \( M = 1 \),

\[
Z(x) = \ln \left( \frac{x}{1-x} \right), \tag{70}
\]

and hence

\[
\lambda_{1}^{p} = 1 + \frac{\lambda_{1}^{p} - 1}{2} \left[ 1 - \tanh \left( \frac{1-\lambda_{1}^{p}}{2-K\xi} \right) \right]. \tag{71}
\]

Similarly for \( M = 2 \),

\[
Z(x) = 2 \ln \left( \frac{x}{1-x} \right) - \frac{1}{x} + \frac{1}{1-x}. \tag{72}
\]

The solutions corresponding to \( M = 1 \) and \( M = 2 \) are shown in Fig. 9, and compared to the numerical quadrature of Eq. (46). The closed form and numerical solutions are in good agreement, lending credence to the simplifying assumptions proposed here.

VI. PARAMETRIC ANALYSIS

In this section, the effects of material parameters on the shock structure are analyzed, with special emphasis on parameters characterizing the material rate sensitivity. An important question is how to relate experimental data (in particular, shock profiles) to the rate sensitivity characteristics. A key parameter associated with the shock structure is the
characteristic width defined by the relationships (57) or (59). Expressions (61) and (62) will not be considered since they give results very close to, respectively, Eqs. (57) and (59). Note first that the term \((3C/2)(1-\lambda_1^{\sigma})\) is entirely determined by the compression curve \(C_\gamma: \lambda_1 \rightarrow \sigma(\lambda_1)\) and by the loading amplitude (characterized by \(\lambda_1\) or by \(\sigma^{-}\)). Consequently, this term is not affected by the material strain rate sensitivity, as \(C_\gamma\) does not depend on the viscous response of the material. Indeed, viscous effects are embedded in the term \(\max(\lambda_1^{\sigma})(\Phi_\sigma(\lambda_1^{\sigma}))\) with

\[
\Phi_\sigma(\lambda_1^{\sigma}) = bNC_\gamma \left( \frac{1}{T_1^{\sigma}} \right)^{h}.
\]

Using Eq. (59), the characteristic width can be written as

\[
w^{\sigma} = \frac{3C}{2} \frac{1 - \lambda_1^{\sigma}}{\lambda_1^{\sigma}bNC_\gamma(T_1^{\sigma})^{M}}.
\]
Note that the value $h = 4$ reported by Swegle and Grady to be the best fit of experimental results for aluminum, corresponds to $M = 1.78$ with Eq. (75a) and $M = 1.81$ with Eq. (75b). Note also that the coefficients on the right hand side of Eq. (75) may depend upon other material parameters (elastic constants, etc.). The parameter $T^{*\alpha}$ does not affect $h$ and $h_p$, as illustrated by Fig. 13 where $\Delta \sigma$ versus $\max|\varepsilon|$ diagrams are shown for different values of $T^{*\alpha}$. To simplify the drawing, the dependence with respect to $\max|\varepsilon|$ is not represented, since the results are similar.

In Fig. 14, $\Delta \sigma$ is represented in terms of $\max|\varepsilon|$ for various values of $M$ and $T^{*\alpha}$. For the fixed value $T^{*\alpha}_1 = 1.6$ MPa, a family of straight lines is obtained when $M$ is varied; these lines intersect at the point $I$. Similarly, the family of lines associated with $T^{*\alpha}_1 = 100$ MPa intersect at point $J$.

Note that for a given value of $M$, a family of parallel lines is obtained when $T^{*\alpha}_1$ varies as seen in Fig. 13. The exponent $h$ also depends on material parameters other than $M$, namely those which affect the shape of the isothermal compression curve. The values of the second order elastic constants $a_4$, $a_5$, $a_6$ have an important effect on $h$ and $h_p$. For the material characterized in Table I, it was found that $h = 3.92$ and $h_p = 3.98$. By multiplying the values of $a_4$, $a_5$, $a_6$ given in Table I by 0.5, 2, and 10, one has, respectively $(h = 4.15, h_p = 4.20)$, $(h = 3.65, h_p = 3.73)$, and $(h = 3.02, h_p = 3.10)$. The effect of the second order elastic constants on $h$ is illus-

FIG. 11. Variation of the characteristic shock width $w_1$ with respect to the reference stress $T^*_1$, for the stress amplitude $\sigma^- = 3.7$ GPa and two values of the rate sensitivity parameter $M$. In this log–log diagram, the slope is given by $M$.

\[ h = 1.87M + 0.620. \]  

(75b)

FIG. 12. Linear dependence of $h_p$ and $h$ [exponent of $\Delta \sigma$ in the power laws Eq. (64)] with respect to the rate sensitivity parameter $M = 1/m$. The results are very well represented by $h_p = 1.87M + 0.663$ and $h = 1.87M + 0.620$.

FIG. 13. Stress jump across the shock $\Delta \sigma = \sigma^- - \sigma^+$ vs $\max|\varepsilon|$, for various values of $T^*_1$. The slope $1/h_p$ is shown to be independent upon $T^*_1$. Similarly $h$ is independent of $T^*_1$. Material parameters are those of Table I, except for $T^*_1 = 2$ and 4 MPa.

FIG. 14. Stress jump across the shock $\Delta \sigma = \sigma^- - \sigma^+$ vs $\max|\varepsilon|$, for various values of the viscosity parameters $M$ and $T^*_1$. When $M$ has a fixed value, and $T^*_1$ is varied, parallel straight lines are obtained as in Fig. 13. For $T^*_1 = 1.6$ MPa, a family of lines is obtained when $M$ varies; these lines intersect at $I$. Similarly, for $T^*_1 = 100$ MPa, a family of lines is obtained which intersect at $J$. A given choice of parameters $M$ and $T^*_1$, corresponds to a line with slope $1/h_p$. This diagram illustrates the dependence of $h_p$ upon $M$ and $T^*_1$. Similar results are obtained when considering $h$. 

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shown in Fig. 14, the slope and the level of this plot is for a given shock strength $T_1$, while for a given shock strength $T_1$, there exist an infinity of pairs $(M, T_1^p)$ producing the same width. Considering two such couples, $(M, T_1^p)$ and $(M', T_1'^p)$, they are related by the relationship

$$\left(\frac{\bar{T}}{T_1^p}\right)^M = \left(\frac{\bar{T}}{T_1'^p}\right)^{M'},$$

where $\bar{T} = |\tau + \sigma|^2$ is evaluated on the Rayleigh line for the value $\lambda_1^p$ of the plastic stretch. Therefore, calibrating the shock width does not provide a unique choice for the viscous parameters. However, $(M, T_1^p)$ can be evaluated in an adequate way by calibration with the $(\max|\dot{e}|, \Delta\sigma)$ diagram. As shown in Fig. 14, the slope and the level of this plot is controlled, respectively by $M$ and $T_1^p$. A further check is provided by the consideration of particle velocity versus time profiles as in Fig. 5. In Fig. 16, particle velocity versus time profiles predicted for $(M, T_1^p) = (2, 2.23$ MPa) are considered and it appears that experimental data are indeed well reproduced. This illustrates again the fact that the sole consideration of these shock profiles is not sufficient to provide a unique choice of the viscous parameters.

However, as discussed before, the value of the exponent $h$ in Eq. (64b) provides an additional constraint for selecting appropriate values of $(M, T_1^p)$. One obtains $(h = 4.35, h_p = 4.40)$ for $(M, T_1^p) = (2, 2.23$ MPa). The result $(h = 3.92, h_p = 3.98$) found for the values listed in Table I, namely $(M, T_1^p) = (1.78, 1.6$ MPa) is in better agreement with the value $h = 4$ estimated by Swegle and Grady for aluminum.

VII. SUMMARY

Steady plastic shock waves observed in metals during plate impact experiments have been analyzed. Considering shocks of moderate amplitude, a simplified framework was adopted in which the process can be considered as quasi-
isentropic. Thermal effects have been evaluated and are shown to be negligible for the range of stress amplitudes considered here. Results were illustrated by choosing aluminum as the model material. Following Clifton,\textsuperscript{17} the material response of aluminum is characterized via two different tests: elastic–plastic characteristics are taken from an isothermal tensile test and second order elastic constants are given by ultrasonic measurements.

Using a Lagrangian setting appropriate for large deformation, an analytical solution of steady plastic shocks has been formulated. This solution provides a clear insight concerning the effects of material parameters and loading conditions on the shock profile. In particular, an explicit formula has been obtained for the characteristic shock width. The effect of viscous like parameters governing the viscoplastic flow such as strain rate sensitivity and flow stress resistance on the shock profile have been analyzed. Conversely, the extraction of these viscous parameters from particle velocity profile measurements has been discussed. Hence, shock wave experiments could provide valuable information on the viscous response of materials subjected to very high loading rates, in addition to their traditional role of establishing their Hugoniot relations.

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**APPENDIX A: SECOND ORDER ELASTIC CONSTANTS**

The function $F_1$ introduced in Eq. (11) is defined as\textsuperscript{17}

$$
F_1(e_1^i, e_2^i, e_3^i) = a_1 + 2a_2(e_1^i + e_2^i + e_3^i) + a_3(e_1^i + e_2^i + e_3^i)
+ 3a_4(e_1^i + e_2^i + e_3^i)^2 + ... + a_6[(e_1^i + e_2^i + e_3^i)]
+ a_6 e_2^i e_3^i. \quad (A1)
$$

$F_2$ appearing in Eq. (12) is obtained from $F_1$ by interchanging $e_1^i$ and $e_2^i$, and by changing $a_1$ into $\tilde{a}_1$

$$
F_2(e_1^i, e_2^i, e_3^i) = F_1(e_2^i, e_1^i, e_3^i). \quad (A2)
$$

If initial stresses are equal to zero (which is the case in the present article), one has

$$
a_1 = 0, \quad \tilde{a}_1 = 0. \quad (A3)
$$

Because of axisymmetry, $F_1$ and $F_2$ can be evaluated by setting $e_3^i = e_2^i$. Therefore one can write

$$
F_1(e_1^i, e_2^i) = a_1 + A_1 e_1^i + A_2 e_1^2 + B_1 e_2^i + B_2 e_2^2 + D e_1^i e_2^i, \quad (A4)
$$

where

$$
A_1 = 2a_2, \quad A_2 = 3a_4,
B_1 = 4a_4 + 2a_3, \quad B_2 = 12a_4 + 5a_5 + a_6. \quad (A5)
$$

$D = 12a_4 + 4a_5$.

The values of the coefficients $a_i$ are given in Table I for aluminum; $a_1$ and $\tilde{a}_1$ are related to the initial stresses. The second order thermoelastic constants $a_4$, $a_5$, and $a_6$ have been identified with ultrasonic measurements.\textsuperscript{17} Note that expression (A1) of $F_1$ is only valid for isentropic transformations. A general expression of $F_1$ and $F_2$, including entropic terms, is given by Clifton.\textsuperscript{17}

**APPENDIX B: DERIVATIVE OF $g_R$**

The derivative of $g_R$ with respect to $\lambda_1^p$ can be written as

$$
g_R'(\lambda_1^p) = 1 + f(X) + (1/2)(1 + X)f'(X), \quad (B1)
$$

where

$$
X = (\lambda_1^p)^{1/2} - 1, \quad (B2)
$$

$$
f'(X) = -\frac{B'}{2A_2} + \frac{1}{4A_2} \frac{2BB' - 4A_2G'}{\sqrt{B^2 - 4A_2G}}, \quad (B3)
$$

$$
B' = dB/dX = D - 4C^2(1 + X), \quad (B4)
$$

$$
G' = B_1 + 2B_2X + 2(C^2\lambda^+ + \sigma^+ / \rho_0)(1 + X) - 4C^2(1 + X)^3. \quad (B5)
$$

**APPENDIX C: EFFECT OF TEMPERATURE**

The variation of temperature $\Delta \theta$ is due to the superposition of two contributions. The compression of a solid produces a temperature rise $\Delta \theta_{\text{Ent}}$, similar to the heating of a gas under adiabatic compression, which is the thermoeelastic effect (or entropic effect). For isentropic conditions...
where \( e_1 \) and \( e_2 \) are the elastic strains defined in Eqs. (13) and (14), and \( b_1, b_2, b_3 \) are thermo–elastic constants given in Table I.\(^{17}\) The second contribution is related to the dissipation of mechanical work related to the physics of plastic deformation. Considering that a proportion \( b_T Q \) (Taylor–Quinney coefficient) of the plastic work \( W_p \) is dissipated into heat, the following contribution to the temperature increase is obtained:

\[
\Delta \theta_{\text{plast}} = b_T Q W_p / \rho_0 C_p. \tag{C2}
\]

where \( C_p \) is the specific heat at constant pressure. The rate of plastic work is given by\(^{17}\)

\[
\dot{W}_p = -\frac{4}{3} \lambda_1 \tau \phi. \tag{C3}
\]

Finally, by superposition the temperature variation with respect to the initial ambient temperature \( \theta_0 \) is obtained

\[
\Delta \theta = \theta - \theta_0 = \Delta \theta_{\text{ent}} + \Delta \theta_{\text{plast}}. \tag{C4}
\]

In Fig. 17, \( \Delta \theta \) is plotted at the rear of the plastic shock, state \(-\), in terms of the stress amplitude \( \sigma^- \), for aluminum with material properties given in Table I. For \( \sigma^- = 0.473 \) MPa, the plastic shock does not exist; \( \Delta \theta = 2.46 \) K is the temperature increase (solely due to the thermoelastic effect) at the rear of the elastic precursor, state \(+\). For \( \sigma^- = 9 \) GPa, one obtains \( \Delta \theta = 70 \) K. This temperature increase does not affect the mechanical response of aluminum \( (T_{\text{melt}} = 933 \) K) significantly. Therefore, having neglected the effect of the temperature in the present model is justified for shocks of moderate amplitude.

\(^{5}\) W. Band, J. Geophys. Res. 65, 695 (1960).